A new formulation of the effective theory for heavy particles

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Abstract

We derive the effective theories for heavy particles with a functional integral approach by integrating away the states with high velocity and with high virtuality. This formulation is non-perturbative and has a close connection with the Wilson renormalization group transformation. The fixed point hamiltonian of our transformation coincides with the static hamiltonian and irrelevant operators can be identified with the usual $1/M$ corrections to the static theory. No matching condition has to be imposed between the full and the static theory operators with our approach. The values of the matching constants come out as a dynamical effect of the renormalization group flow.

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1 Introduction

Effective theories for heavy particles are a powerful tool to study heavy flavor physics. In recent years, many properties of the spectrum and of the decays of heavy hadrons have been understood [1]. The idea underlying the effective theory approach is that typical (i.e. after renormalization) momentum transfers between quarks and gluons in a hadron are of the order of the QCD scale $\Lambda_{QCD} \sim 300 \ MeV$. As a consequence, relativistic processes for a "heavy quark" Q, with a mass $M \gg \Lambda_{QCD}$, are suppressed. The heavy quark is almost at rest and on-shell, and chromomagnetic interactions are weak [2]. Furthermore, the creation of heavy $Q\overline{Q}$ pairs may be neglected.

In general, we consider processes where *heavy* particles are subjected to *soft* interactions. By 'soft' we mean that the energy transfer ϵ and the momentum transfer $\vec{q}$ to the heavy particle H are much less than its mass M:

$$| \, \epsilon \, |, \ \ | \, \vec{q} \, | \ \ \ll \ M. \tag{1}$$

Before the scattering H is on-shell and, let us say, at rest. After soft interactions, H will remain essentially on-shell and will acquire a very small velocity. The relevant states for the dynamics of H will be those with 4-momentum around the rest momentum

$$r \ = \ (M, \vec{0}). \tag{2}$$

The effective theories are derived generally by means of an expansion in k/M of the lagrangian [3], where k is the residual momentum defined as

$$k \ = \ p - r, \tag{3}$$

and p is the momentum of the heavy particle. This expansion is of classical nature and the effects of quantum fluctuations are taken into account imposing a set of matching conditions to the effective theory operators [4].

We reformulate the effective theories for heavy particles with a functional integral approach [5]. We start with a field theory with a cut-off Λ much larger than the heavy particle mass,

$$\Lambda \ \gg \ M, \tag{4}$$

and we integrate away:

i) the states with a large velocity, for which the spatial momentum $| \, \vec{p} \, | \sim M$ or greater than that;

ii) the states which are highly virtual, i.e. the states with an invariant mass $p^2 \ll M^2$ or $p^2 \gg M^2$.

We leave in the effective action S_{eff} the states with momenta p inside a small neighborhood of the on-shell momentum of eq. (2) (see fig.1):

$$(p_0 - M)^2 \; + \; \vec{p}^{\,2} \quad \leq \quad \Lambda_E^2, \tag{5}$$

where Λ_E is the ultraviolet cut-off of the effective theory, well below the heavy particle mass:

$$\Lambda_E \quad \ll \quad M. \tag{6}$$

In this way we confine ourselves to the low-energy domain. The effective action iS_{eff} is defined as:

$$\exp\{ \; iS_{eff}[\phi(k); 0 < k^2 < \Lambda_E^2] \; \} \doteq \tag{7}$$
$$\int \prod_{\Lambda_E^2 < k^2 < \Lambda^2} d\phi(k) \; \exp\{ \; iS[\phi(l); 0 < l^2 < \Lambda^2] \; \}.$$

There exists a different definition of S_{eff} with respect to that one given in eq. (7). We may also consider the construction of the effective action for heavy particles as a scale transformation. The system is viewed at progressively larger scales with respect to the Compton wavelength of the heavy particle, $\lambda_C/2\pi = 1/M$. We include therefore in the definition of the effective action, in addition to the integration of the high-energy modes, also a rescaling of the residual momenta and of the fields. This second formulation has a closer relation with the Wilson renormalization group (RG) transformation [6] and consists of the following steps:

1) We lower the ultraviolet cut-off of the effective action by a factor s, i.e. from Λ_E to Λ_E/s. The modes with momenta between Λ_E and Λ_E/s are integrated:

$$\exp\{ \; iS_{eff}[\phi(k); 0 < k^2 < (\Lambda_E/s)^2] \; \} \doteq \tag{8}$$
$$\int \prod_{(\Lambda_E/s)^2 < k^2 < \Lambda_E^2} d\phi(k) \; \exp\{ \; iS[\phi(l); 0 < l^2 < \Lambda_E^2] \; \}.$$

2) The (residual) momentum is rescaled according to:

$$k' = sk. \tag{9}$$

Notice that one does not rescale the energy of the heavy quark, but the energy minus the mass.

3) The heavy particle field ϕ is rescaled in such a way that the kinetic operator takes a unit coefficient:

$$\phi'(k') = \zeta \, \phi(k). \tag{10}$$

This transformation can be iterated many times. At each step we lower the cut-off Λ_E by a factor s, i.e. from Λ_E to Λ_E/s, from Λ_E/s to Λ_E/s^2, etc., and we generate a sequence of effective actions S_{eff}, $S_{eff}^{(1)}$, $S_{eff}^{(2)}$, ... Expanding the effective action S_{eff} in a basis of local operators O_n,

$$S_{eff} = \sum_n c_n O_n, \tag{11}$$

where c_n are given coefficients, the RG transformation has the following representation:

$$c_n^{(l+1)} = f_n(c_1^{(l)}, c_2^{(l)}, ..., c_k^{(l)}, ...; s) \quad n = 1, 2, 3, ..., k, ... \quad . \tag{12}$$

These equations are transformed into differential equations by setting $s = 1 + \delta s$ with $\delta s \ll 1$, and expanding up to first order in δs.
The fixed point action S^* (if any) describes particles which can change their momenta with respect to the rest momentum in eq. (2) by a tiny amount only, i.e. it describes particles which are essentially static. S^* therefore must be identified with the static action. A fixed point is a solution of:

$$c_n^* = f_n(c_1^*, c_2^*, ..., c_k^*, ...; s) \quad\quad n = 1, 2, 3, ..., k, ... \quad . \tag{13}$$

Finding a fixed point for our RG transformation can be considered as a rigorous proof of the existence of the static limit in quantum field theory.

Let us make some remarks about the differences between the formulation specified by eqs. (8-10) and the usual one.

With our formulation we do not deal with lagrangians but directly with the functional integral: quantum fluctuations are taken into account from

the very beginning. In the usual approach, one performs an expansion in $1/M$ of the (relativistic) lagrangian $\mathcal{L}$ for a heavy particle to derive the static lagrangian $\mathcal{L}_{st}$. Tree level amplitudes computed with $\mathcal{L}$ and with $\mathcal{L}_{st}$ coincide at order $(1/M)^0$, implying no need of lowest order matching. Loop diagrams are sensitive to high energy physics and turn out to have a different value in the full and in the static theory. This implies that quantum fluctuations spoil tree level matching, and one is forced to redefine the matching constants in order to include their effects. With our approach, there is not any matching condition to impose: one has simply to integrate away the off-shell modes and look at the modifications of the coefficients of the operators.

Unlike classical physics or quantum mechanics, the static approximation is singular in quantum field theory. The reason is that the heavy particle mass M acts as a (physical) ultraviolet cut-off in many processes. Logs of M, for example, appear in one loop corrections to heavy currents [4]. In the static theory, the heavy particle mass is removed and the resulting divergence is regulated by Λ_E. The cut-off of the effective theory, therefore, does not regulate only the divergences related to mass, field and coupling constant renormalization, but also those ones induced by the limit $M \rightarrow \infty$. With our approach, the singular behaviour of the static approximation is revealed by an anomalous scaling as we lower progressively Λ_E.

Ordinary RG transformation is a tool to describe *soft* interactions of *light* particles in the framework of a field theory. In this case the states with large *full* energy and with large 3-momenta are integrated away and the effective action S_{eff} describes the dynamics of light particles with momenta inside a small neighborhood of the null momentum (see fig.2)

$$n = (0, \vec{0}). \tag{14}$$

We see therefore that our construction of the effective action for heavy particles specified by eqs. (8-10) closely parallels Wilson's RG transformation. The difference is that we integrate away the states with momenta far from the rest momentum of a heavy particle instead of the null momentum (see ref.[7] where an intuitive discussion on the above point is presented).
This paper is organized as follows.

In Sect. 2 we compute the RG transformation (8-10) in the free case. This is the simplest case (exactly solvable) and will serve as an illustration of the basic ideas and of the formalism.

In Sect. 3 we construct the RG transformation for a simple (interacting) model, and we discuss the general approximations involved. A perturbative evaluation of the RG transformation and of its fixed points is also presented.

In Sect. 4 we compute the effective hamiltonian as given by eq. (7) - i.e. without rescaling - for the same model, and we discuss the relation between the original and the effective hamiltonian. A perturbative evaluation of the matching constants is presented and the relation with ordinary matching theory is treated.

Sect. 5 contains the conclusions of our analysis.

2 Free case

Let us consider the construction of the effective action in a very simple case, a free scalar of mass M in Minkowski space. The case of fermions will be discussed in a subsequent work. The action is given by:

$$iS = i \int_0^\Lambda \frac{d^D p}{(2\pi)^D} \ \Phi^\dagger(p) \ [p^2 - M^2 + i\epsilon] \ \Phi(p). \tag{15}$$

Since what is small is not the energy of the heavy particle, but the energy minus the mass, it is convenient to express the action in terms of the residual momentum k. One has:

$$iS = i \int_0^\Lambda \frac{d^D k}{(2\pi)^D} \ \phi^\dagger(k) \ [k_0 + \frac{k^2}{2M} + i\epsilon] \ \phi(k), \tag{16}$$

where we have defined an 'effective field' ϕ such that:

$$\phi(k) \ \doteq \ \sqrt{2M} \ \Phi(r + k). \tag{17}$$

The difference between the domains of integration of p and k has been neglected because of the condition stated in eq. (4). In the free case, the functional integration is trivial because there is no coupling between high-momentum and low-momentum modes, and gives an effective action of the same form as the original one, with a smaller cut-off Λ_E:

$$iS_{eff} = i \int_0^{\Lambda_E} \frac{d^D k}{(2\pi)^D} \ \phi^\dagger(k) \ [k_0 + k^2/2M + i\epsilon] \ \phi(k). \tag{18}$$

In order to understand the relevance of the operators entering in S_{eff}, let us compute the RG transformation as defined by eqs. (8-10).

1) Integrating the modes with momenta between Λ_E and Λ_E/s, the effective action becomes:

$$iS'_{eff} = i \int_0^{\Lambda_E/s} \frac{\mathrm{d}^D k}{(2\pi)^D} \, \phi^\dagger(k) \, [k_0 + k^2/2M + i\epsilon] \, \phi(k). \qquad (19)$$

plus a constant which does not affect the value of the correlation functions.

2) Rescaling the momenta according to eq. (9), the effective action reads:

$$iS'_{eff} = is^{-(D+1)} \int_0^{\Lambda_E} \frac{\mathrm{d}^D k'}{(2\pi)^D} \, \phi^\dagger(k'/s) \, [k'_0 + \frac{1}{s}\frac{k'^2}{2M} + i\epsilon] \, \phi(k'/s). \qquad (20)$$

3) Rescaling the field ϕ in such a way that the kinetic operator takes a unit coefficient,

$$\phi'(k') = s^{-(D+1)/2}\phi(k'/s), \qquad (21)$$

the effective action looks finally:

$$iS'_{eff} = i \int_0^{\Lambda_E} \frac{\mathrm{d}^D k}{(2\pi)^D} \phi'^\dagger(k') \, [k'_0 + \frac{1}{s}\frac{k'^2}{2M} + i\epsilon] \, \phi'(k'). \qquad (22)$$

We iterate now the transformation many times, say n times; each iteration the term with k^2 is multiplied by a factor $1/s$ and its coefficient reduces exponentially with n. There exists therefore a fixed point action S^*, which is given by:

$$iS^* = i \int_0^{\Lambda_E} \frac{\mathrm{d}^D k}{(2\pi)^D} \phi^\dagger(k) \, [k_0 + i\epsilon] \, \phi(k). \qquad (23)$$

This is the familiar static action. Note that the mass M of the heavy particle disappears in the fixed point action S^*. In other words, the actions of heavy particles with different masses lie in the same universality class with respect to the RG transformation specified by eqs. (8-10). This is the well-known flavor symmetry of the static theory [8]. The term $k^2/2M$ is an irrelevant operator with respect to our RG transformation. For high level computations this operator may be introduced in the dynamics as an insertion, by means of an expansion of the form:

$$e^{iS_{eff}} = e^{iS^*} \left[1 + i \int_0^{\Lambda_E} \frac{\mathrm{d}^D k}{(2\pi)^D} \phi^\dagger(k)\frac{k^2}{2M}\phi(k) + O(1/M^2)\right]. \qquad (24)$$

According to the short-distance expansion (24), the heavy scalar propagator is given by:

$$i\Delta(k) = \frac{i}{k_0 + i\epsilon} + \frac{i}{k_0 + i\epsilon}\left(i\frac{k_0^2}{2M} - i\frac{\vec{k}^2}{2M}\right)\frac{i}{k_0 + i\epsilon} + O(1/M^2) \quad (25)$$

$$= \frac{i}{k_0 + i\epsilon}\left(1 - \frac{k_0}{2M}\right) + \frac{i}{k_0 + i\epsilon}\frac{-i\vec{k}^2}{2M}\frac{i}{k_0 + i\epsilon} + O(1/M^2).$$

The round bracket contains a contact term, which is the effect of order $1/M$ of the antiparticle pole, i.e. of the states with a very high virtuality. This term can be reabsorbed with a redefinition of the operators containing the heavy field ϕ:

$$\phi \rightarrow \left(1 - i\frac{\partial_0}{4M}\right)\phi. \quad (26)$$

The second term of the last member in eq. (25) is the familiar kinetic energy operator. We conclude therefore that the irrelevant operators with respect to our RG transformation coincide with the operators of order $1/M$ in the expansion of the lagrangian.

The RG transformation constructed above can be easily extended to describe infrared interactions of heavy particles with a fixed velocity v^μ [8, 9]. The only difference is that in the latter case we must leave in the effective action the states with momenta in a small neighborhood of the momentum Mv instead of the rest momentum of eq. (2).

3   A simple model

We present in this section the construction of the effective hamiltonian in an interacting case. We consider a simple model, in euclidean space, whose dynamics is determined by the hamiltonian:

$$\mathcal{H} = \partial_\mu\Phi^\dagger\partial_\mu\Phi + M^2\Phi^\dagger\Phi + \frac{1}{2}\partial_\mu a\partial_\mu a + \frac{1}{2}m^2 a^2 + g\Phi^\dagger\Phi a, \quad (27)$$

where Φ is a heavy scalar field with mass M and a is a light scalar field with mass m. This theory is superrenormalizable in four dimensions $D = 4$, because the coupling g has the dimension of a mass, but this does not matter for the following considerations. We do not deal with the continuum limit of

the model (i.e. with the limit $\Lambda \to \infty$) but rather with its infrared properties. This is indeed a model for QED at small energies compared to the electron mass.

We are interested in the soft interactions between these particles; we introduce therefore an effective theory where we integrate away the states with high-energy for the light particle and the states far from the mass-shell state of eq. (2) for the heavy particle. We select a cut-off Λ_E so that:

$$m^2 \ll \Lambda_E^2 \ll M^2. \tag{28}$$

Let us compute the RG transformation as specified by eqs. (8-10).

1) After making a shift like in eq. (16), the integration of the high energy modes is realized according to:

$$\exp\left\{ - H'_{eff}[\ \phi(k), a(l); 0 < k^2, l^2 < (\Lambda_E/s)^2]\ \right\} =$$

$$\int \prod_{(\frac{\Lambda_E}{s})^2 < k^2, l^2 < \Lambda_E^2} d\phi(k) da(l) \exp\left\{ - \int_0^{\Lambda_E} \frac{d^D k}{(2\pi)^D} \phi^\dagger(k)\left[ik_4 + \delta M + \frac{k^2}{2M}\right]\phi(k)\right.$$

$$- \int_0^{\Lambda_E} \frac{d^D l}{(2\pi)^D}\ a(-l)\ [l^2 + m^2]\ a(l) +$$

$$\left. - \lambda \int_0^{\Lambda_E} \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\frac{d^D k_3}{(2\pi)^D}(2\pi)^D \delta(k_1 - k_2 - k_3)\phi^\dagger(k_1)\phi(k_2)a(k_3)\right\}, \tag{29}$$

where $\lambda = g/2M$ is a dimensionless coupling constant in $D = 4$, and a bare mass term δM has been introduced in the heavy particle lagrangian.
The integrated modes induce non-local couplings in the effective hamiltonian, as well as interactions between an arbitrary number of particles. H'_{eff} therefore is non local and non polynomial even though the original hamiltonian is renormalizable:

$$H'_{eff} = \int_0^{\Lambda_E/s} \frac{d^D k}{(2\pi)^D}\ \phi^\dagger(k)\ \left[ik_4 + \delta M + \frac{k^2}{2M} - \Sigma_H(k)\right]\ \phi(k)$$

$$+ \int_0^{\Lambda_E/s} \frac{d^D l}{(2\pi)^D}\ a(-l)\ \left[l^2 + m^2 - \Sigma_L(l)\right]\ a(l) +$$

$$+ \int_0^{\Lambda_E/s} \frac{d^D k_1}{(2\pi)^D}...\frac{d^D k_3}{(2\pi)^D}(2\pi)^D \delta(k_1... - k_3)\left[\lambda - V^a_{\phi\phi}(k_1, k_2, k_3)\right]\phi^\dagger(k_1)\phi(k_2)a(k_3)$$

$$+ \int_0^{\Lambda_E/s} \frac{d^D k_1}{(2\pi)^D}...\frac{d^D k_4}{(2\pi)^D}(2\pi)^D \delta(k_1... - k_4)V^{aa}_{\phi\phi}(k_1, ..., k_4)\phi^\dagger(k_1)\phi(k_2)a(k_3)a(k_4)$$

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$$+ \quad \dots \quad , \tag{30}$$

where we have separated the tree level coupling from the loop correction in the $\phi\phi a$ vertex, i.e. $V_{\phi\phi}^a$ starts at order λ^3.

An exact evaluation of H'_{eff} is clearly an hopeless task and some approximation is needed. Assuming a basis of local operators, this involves a truncation of the operator series generated by eq. (29). The justification of this approximation relies on scale considerations and will be discussed in 2).

2) Rescaling the full momenta of the light particle and the residual momenta of the heavy particle according to eq. (9), brings the effective hamiltonian into the form:

$$H''_{eff} = \int_0^{\Lambda_E} \frac{d^D k'}{(2\pi)^D} \ s^{-(D+1)} \phi^\dagger(\frac{k'}{s}) \left[ik'_4 + s\delta M + \frac{1}{s}\frac{k'^2}{2M} - s\Sigma_H(\frac{k'}{s})\right] \phi(\frac{k'}{s})$$

$$+ \int_0^{\Lambda_E} \frac{d^D l'}{(2\pi)^D} s^{-(D+2)} \ a(-l'/s) \left[l'^2 + s^2 m^2 - s^2\Sigma_L(l'/s)\right] a(l'/s) +$$

$$+ \int_0^{\Lambda_E} \frac{d^D k'_1}{(2\pi)^D} \cdots \frac{d^D k'_3}{(2\pi)^D} (2\pi)^D \delta(k'_1 \dots - k'_3) s^{-2D} \left[\lambda - V_{\phi\phi}^a(\frac{k'_1}{s}, \frac{k'_2}{s}, \frac{k'_3}{s})\right] \phi^\dagger(\frac{k'_1}{s}) \cdots a(\frac{k'_3}{s})$$

$$+ \int_0^{\Lambda_E} \frac{d^D k'_1}{(2\pi)^D} \cdots \frac{d^D k'_4}{(2\pi)^D} (2\pi)^D \delta(k'_1 \dots - k'_4) s^{-3D} V_{\phi\phi}^{aa}(\frac{k'_1}{s}, \dots, \frac{k'_4}{s}) \phi^\dagger(\frac{k'_1}{s}) \cdots a(\frac{k'_4}{s})$$

$$+ \quad \dots \quad . \tag{31}$$

The original limit of integration, Λ_E, has been restored.
Let us discuss now the main approximation in the evaluation of H_{eff}. We consider first the free field case ($\lambda = 0$). As it stems from eq. (31), the mass dimensions of the fields $\phi(k)$ and $a(l)$ are given respectively by:

$$-\frac{D+1}{2}, \quad -\frac{D+2}{2}. \tag{32}$$

The dimension of a derivative operator is one. Then, the dimension of a composite operator O (integrated over space) containing n_ϕ ϕ-fields , n_a a-fields and n_∂ derivatives is given by:

$$d_O = -D + n_\phi\frac{D-1}{2} + n_a\frac{D-2}{2} + n_\partial. \tag{33}$$

Each RG iteration, the operator O is multiplied by the factor s^{-d_O}. Therefore, an operator with a positive mass dimension, $d_O > 0$, has a weight which

reduces exponentially with the number of iterations, and can be neglected at leading level. In the case with interactions ($\lambda \neq 0$), we assume that their effects are small, in the sense that anomalous scaling does not overwhelm the canonical one. Also in the latter case, therefore, we can neglect the operators with a positive mass dimension.

The only relevant operators (i.e. with $d_O \leq 0$) are those already present in the original hamiltonian,

$$\phi^\dagger \phi, \quad \phi^\dagger k_4 \phi, \quad a^2, \quad al^2 a, \quad \phi^\dagger \phi a, \tag{34}$$

plus

$$\phi^\dagger(k) k_i \phi(k) \qquad \text{for} \ \ i = 1, 2, 3. \tag{35}$$

The latter are indeed the operators which enter in the Georgi lagrangian [8]. We will see in section (3.1) that they are not induced in H_{eff} for the static case in a regularization preserving parity.

At the leading level, therefore, the RG transformation amounts to a renormalization of the original couplings only:

$$
\begin{aligned}
H''_{eff} &\cong \int_0^{\Lambda_E} \frac{d^D k'}{(2\pi)^D} s^{-(D+1)} \phi^\dagger(\frac{k'}{s}) \left[\left(1 + i \left(\frac{\partial \Sigma_H}{\partial k_4}\right)_0\right) ik'_4 + s\left(\delta M - \Sigma_H(0)\right) \right] \phi(\frac{k'}{s}) \\
&+ \int_0^{\Lambda_E} \frac{d^D l'}{(2\pi)^D} s^{-(D+2)} a(-\frac{l'}{s}) \left[\left(1 - \left(\frac{\partial \Sigma_L}{\partial l^2}\right)_0\right) l'^2 + s^2\left(m^2 - \Sigma_L(0)\right) \right] a(\frac{l'}{s}) + \\
&+ \int_0^{\Lambda_E} \frac{d^D k'_1}{(2\pi)^D} ... \frac{d^D k'_3}{(2\pi)^D} (2\pi)^D \delta(k'_1 ... - k'_3) s^{-2D} \left[\lambda - V^a_{\phi\phi}(0,0,0) \right] \phi^\dagger(\frac{k'_1}{s}) \phi(\frac{k'_2}{s}) a(\frac{k'_3}{s}).
\end{aligned}
\tag{36}
$$

3) We rescale now the heavy and the light particle fields in such a way that the kinetic operators $\phi^\dagger(k) ik_4 \phi(k)$ and $a(-l)l^2 a(l)$ respectively, take unit coefficient:

$$
\begin{aligned}
\phi'(k') &= s^{-(D+1)/2} \left[1 + i \left(\frac{\partial \Sigma_H}{\partial k_4}\right)_0 \right]^{1/2} \phi(k'/s) \\
a'(l') &= s^{-(D+2)/2} \left[1 - \left(\frac{\partial \Sigma_L}{\partial l^2}\right)_0 \right]^{1/2} a(l'/s). \tag{37}
\end{aligned}
$$

The effective hamiltonian takes finally the form:

$$H_{eff}''' = \int_0^{\Lambda_E} \frac{d^D k'}{(2\pi)^D} \; \phi'^\dagger(k') \left[i k_4' + \delta M' \right] \phi'(k') +$$

$$+ \int_0^{\Lambda_E} \frac{d^D l'}{(2\pi)^D} \; a'(-l') \left[l'^2 + m^{2'} \right] a'(l') +$$

$$+ \; \lambda' \int_0^{\Lambda_E} \frac{d^D k_1'}{(2\pi)^D} \frac{d^D k_2'}{(2\pi)^D} \frac{d^D k_3'}{(2\pi)^D} (2\pi)^D \delta(k_1' - k_2' - k_3') \phi'^\dagger(k_1') \phi'(k_2') a'(k_3'),$$

(38)

where the new coupling λ' and the new mass parameters $\delta M'$ and $m^{2'}$ are given by:

$$\lambda' = s^{2-D/2} \left[1 - \frac{V_{\phi\phi}^a(0,0,0)}{\lambda} \right] \left[1 + i \left(\frac{\partial \Sigma_H}{\partial k_4} \right)_0 \right]^{-1} \left[1 - \left(\frac{\partial \Sigma_L}{\partial l^2} \right)_0 \right]^{-1/2} \lambda$$

$$\delta M' = s \left[1 + i \left(\frac{\partial \Sigma_H}{\partial k_4} \right)_0 \right]^{-1} \left[\delta M - \Sigma_H(0) \right]$$

$$m^{2'} = s^2 \left[1 - \left(\frac{\partial \Sigma_L}{\partial l^2} \right)_0 \right]^{-1} \left[m^2 - \Sigma_L(0) \right].$$

(39)

Eqs. (39) are an approximation of the exact equations (12) for the *RG* flow and, together with (37), represent the leading effects of the *RG* transformation. To proceed, we must compute explicitly the functions entering in eqs. (39). This is done in the next section with perturbative methods. We will search the fixed points of the truncated eqs. (39) in Sect. 3.2.

3.1 Perturbative computation

The construction of the effective hamiltonian presented in Sect. 3 is non-perturbative; it is basically a prescription for the degrees of freedom to integrate away. As such, it does not rely on any Feynman diagram expansion, just like Wilson transformation. In deriving it, we assumed only that the effects of interactions are not so strong to destroy the distinction between relevant and irrelevant operators given by the free theory. This assumption is indeed quite reasonable from the physical viewpoint, and it can only be checked case by case with an exact computation. However, in order to develop an intuitive understanding of the properties of our transformation, and

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to make contact with the usual matching theory, we present in this section a perturbative evaluation of the effective hamiltonian.

Let us sketch the derivation of the Feynman rules for the evaluation of H'_{eff} [10]. We decompose the fields ϕ and a into their low energy and high energy parts:

$$\phi(k) = \phi_L(k) + \phi_H(k), \qquad a(l) = a_L(l) + a_H(l), \tag{40}$$

where the fields with the subscripts 'L' or 'H' contain modes with momenta less than or above Λ_E/s respectively:

$$\begin{aligned} \phi_L(k) &= \phi(k)\,\theta(\Lambda_E/s - k), & \phi_H(k) &= \phi(k)\,\theta(k - \Lambda_E/s) \\ a_L(l) &= a(l)\,\theta(\Lambda_E/s - l), & a_H(l) &= a(l)\,\theta(l - \Lambda_E/s). \end{aligned} \tag{41}$$

We insert then the decomposition (40) in the right hand side of eq. (29). In the bilinear (free) part of the hamiltonian there are no couplings between 'L' and 'H' fields. On the contrary, the trilinear (interaction) term generates any possible couplings between them. We expand the right hand side of eq. (29) in powers of λ and we compute the functional integral, which is only over ϕ_H and a_H. Each power of ϕ_L and a_L is associated to an external leg of the diagrams, while each bilinear in ϕ_H or a_H is associated to an internal line. The external lines of the diagrams have momenta between zero and Λ_E/s while the internal lines have momenta in the range $\Lambda_E/s - \Lambda_E$. Loops are integrated in a region where all the internal lines have momenta between Λ_E/s and Λ_E. The Feynman rules are:

$$\frac{-i}{k_4 - ik^2/2M} \quad : \quad \text{heavy scalar propagator} \tag{42}$$

$$\frac{1}{l^2 + m^2} \quad : \quad \text{light scalar propagator} \tag{43}$$

$$-\lambda \quad : \quad \text{vertex.} \tag{44}$$

Note the asymmetry between the heavy and the light propagator related to the shift in the energy for the massive one; the poles for the heavy particle and the heavy antiparticle are respectively at:

$$k_4 = i(M \mp \sqrt{\vec{k}^2 + M^2}). \tag{45}$$

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By taking the logarithm on both sides of eq. (29), one sees that only connected diagrams contribute to H'_{eff}. Furthermore, one particle reducible diagrams do not contribute to the effective hamiltonian in the low momentum region.

The exact determination of H'_{eff} therefore requires to compute and to sum all the connected diagrams with an arbitrary number of external legs and with generic momenta below Λ_E/s. Loops induce in the effective hamiltonian interactions with an arbitrary number of fields and with an arbitrary number of derivatives. As explained in the previous section, we neglect higher dimension operators since they are irrelevant, and we limit to one loop accuracy.

Let us go then to the explicit computation of the diagrams. From now on we stick to the case $D = 4$. At one-loop level, the one-particle irreducible diagrams are classified by the number of external heavy and light lines, (n_H, n_L). The self-energy of the heavy scalar $(n_H = 2, n_L = 0)$ is given by:

$$\Sigma_H(k) = -i\lambda^2 \int_{D2} \frac{d^4l}{(2\pi)^4} \frac{1}{l_4 - il^2/2M} \frac{1}{(k-l)^2 + m^2}, \tag{46}$$

where $D2$ is the region of the l-space, where both propagators have momenta between Λ_E/s and Λ_E:

$$(\Lambda_E/s)^2 < l^2, \ (k-l)^2 < \Lambda_E^2. \tag{47}$$

Since $m^{2*} = 0$ (see later), we can neglect m^2 in eq. (46).

The self-energy at zero external momentum is given by:

$$\Sigma_H(0) = \frac{\lambda^2}{16\pi^2} 8\Lambda_E \left[1 - \frac{1}{s} + O(\Lambda_E/M) \right]. \tag{48}$$

The first derivative of $\Sigma_H(k)$ with respect to k_4 is given by:

$$\frac{\partial \Sigma_H}{\partial k_4}\Big|_{k=0} = -i\frac{\lambda^2}{16\pi^2} 2 \left[\log s^2 + O(\Lambda_E/M) \right]. \tag{49}$$

We note that the Taylor expansion of the self-energy around the on-shell point $k = 0$ is not singular because the lower cut-off Λ_E/s acts as a regulator for the soft divergences.

First derivatives of $\Sigma_H(k)$ with respect to spatial momentum components k_i vanish due to parity. Therefore, linear operators in the spatial derivatives

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of the form $\phi^\dagger \partial_i \phi$ are not induced in the effective hamiltonian by loop effects (this was implicitly assumed in eq. (36), where we have done a Taylor expansion of $\Sigma_H(k)$ to disentangle the leading terms in H_{eff}).

The non-vanishing second derivatives of Σ_H are given by:

$$\left.\frac{\partial^2 \Sigma_H}{\partial k_4^2}\right|_{k=0}, \quad \left.\frac{\partial^2 \Sigma_H}{\partial k_i^2}\right|_{k=0}, \tag{50}$$

and turn out to be proportional to s/Λ_E. They are related respectively to the irrelevant operators $\phi^\dagger \partial_4^2 \phi$ and $\phi^\dagger \partial_i^2 \phi$.

The self-energy of the light particle $\Sigma_L(l)$ involves a virtual heavy pair and can be neglected at the lowest order in $1/M$ [11]:

$$\Sigma_L(l) = 0. \tag{51}$$

The effects of heavy pairs, together with all the other subleading effects [12], will be treated in a subsequent work.

The vertex correction is given by:

$$
\begin{aligned}
V_{\phi\phi}^a(k, k') &= \lambda^3 \int_{D3} \frac{d^4 l}{(2\pi)^4} \frac{1}{l^2 + m^2} \times \\
&\times \frac{1}{k_4 + l_4 - i(k+l)^2/2M} \frac{1}{k_4' + l_4 - i(k'+l)^2/2M},
\end{aligned} \tag{52}
$$

where k and k' denote respectively the momenta of the incoming and outcoming heavy particle. At zero external momenta:

$$V_{\phi\phi}^a(0, 0) = -\frac{\lambda^3}{16\pi^2}\left[2 \log s^2 + O(\Lambda_E/M)\right]. \tag{53}$$

3.2   Fixed points

Since the first of eqs. (39) for the coupling constant λ is homogenous, there exists a gaussian fixed point:

$$
\begin{aligned}
\lambda^* &= 0 \\
\delta M^* &= 0 \\
m^{2*} &= 0.
\end{aligned} \tag{54}
$$

This fixed point is trivial, and we seek a fixed point near this one (i.e. with $\lambda^* \ll 1$), corresponding to a weakly interacting theory.

In one-loop approximation, the new coupling constant λ' is a function of the old coupling constant λ only, while the new mass parameters $\delta M'$ and $m^{2'}$ depend on the original mass parameters δM and m^2 and on λ:

$$
\begin{aligned}
\lambda' &= f(\lambda) \\
\delta M' &= g(\delta M, \lambda) \\
m^{2'} &= h(m^2, \lambda),
\end{aligned}
\tag{55}
$$

where f, g and h are given functions, which we will compute at one loop level later in this section. We note that the equations of Wilson's RG transformation have the same structure as that in eqs. (55).

The problem is therefore that of finding a fixed point for the coupling constant equation:

$$
\lambda^* = f(\lambda^*).
\tag{56}
$$

Once this has been found, we tune the original mass parameters in such a way that they are unchanged by the transformation; we have simply to solve the equations:

$$
\begin{aligned}
\delta M^* &= g(\delta M^*, \lambda^*) \\
m^{2*} &= h(m^{2*}, \lambda^*).
\end{aligned}
\tag{57}
$$

Unlike the free case, we have $\delta M^* \neq 0$ and $m^{2*} \neq 0$, in order to cancel the contribution to mass renormalization coming from integrated (high energy) modes.

Substituting eqs. (49), (51) and (53) in the coupling constant equation (39), we see that the $\log s^2$ terms cancel, giving:

$$
\lambda' = \lambda,
\tag{58}
$$

plus corrections of order Λ_E/M. The coupling constant therefore does not change with the RG transformation. This occurs because heavy pairs are absent and the model reduces to a free scalar field interacting with fixed sources. We have therefore:

$$
\begin{aligned}
\lambda^* &= \lambda_0 \\
m^{2*} &= 0 \\
\delta M^* &= \frac{\lambda^{*2}}{16\pi^2} 8\Lambda_E.
\end{aligned}
\tag{59}
$$

There exists a fixed point hamiltonian H^* with a coupling equal to the original one. Therefore, if we start with an initial coupling $\lambda_0 \ll 1$ at a scale $\sim M$, we remain in a region of small coupling, where our perturbative computation is trustworthy. We see that the formalism developed has a complete correspondence with physical intuition. We can consider the result (59) as a rigorous proof of the existence of the static limit for our model.

4 Matching

We consider now the construction of the effective hamiltonian as specified by eq. (7), i.e. without the rescaling of the momenta and of the fields. This is indeed the formulation which can be more easily related to the usual approach to the effective hamiltonians and to matching theory. We integrate now the degrees of freedom with momenta between Λ and Λ_E, instead of between Λ_E and Λ_E/s as we have done in the previous section.
At leading level, the effective hamiltonian is given by:

$$
\begin{aligned}
H_{eff} &\cong \int_0^{\Lambda_E} \frac{d^D k}{(2\pi)^D} \phi^\dagger(k) \left[\left(1 + i \left(\frac{\partial \tilde{\Sigma}_H}{\partial k_4} \right)_0 \right) i k_4 + \left(\delta M - \tilde{\Sigma}_H(0) \right) \right] \phi(k) \\
&+ \int_0^{\Lambda_E} \frac{d^D l}{(2\pi)^D} a(-l) \left[\left(1 - \left(\frac{\partial \tilde{\Sigma}_L}{\partial l^2} \right)_0 \right) l^2 + \left(m^2 - \tilde{\Sigma}_L(0) \right) \right] a(l) + \qquad (60) \\
&+ \int_0^{\Lambda_E} \frac{d^D k_1}{(2\pi)^D} ... \frac{d^D k_3}{(2\pi)^D} (2\pi)^D \delta(k_1 - k_2 - k_3) \left[\lambda - \tilde{V}_{\phi\phi}^a(0,0,0) \right] \phi^\dagger(k_1) \phi(k_2) a(k_3),
\end{aligned}
$$

where a tilde indicates the new range of integration.
The functions $\tilde{\Sigma}_H(0)$, $\tilde{\Sigma}_L(0)$, $(\partial \tilde{\Sigma}_H/\partial k_4)_0$, $(\partial \tilde{\Sigma}_L/\partial l^2)_0$ and $\tilde{V}_{\phi\phi}^a(0,0,0)$ contain the leading effects of the integrated modes. We present a one-loop calculation of these functions in Sect. 4.1. Sect. 4.2 contains a discussion and a generalization of the one-loop results. The functions entering in eq. (60) can be related to the perturbative matching constants between the full and the static theory. This will be done in Sect. 4.3.

4.1 Perturbative computation

The Feynman rules are the same as those in Sect. 3.1 with the only difference that now loops are integrated in a region where all the internal lines have momenta between Λ and Λ_E, instead of between Λ_E and Λ_E/s.

The self-energy of the heavy scalar is given by:

$$\tilde{\Sigma}_H(k) = -i\lambda^2 \int \frac{d^4l}{(2\pi)^4} \frac{1}{l_4 - il^2/2M} \frac{1}{(k-l)^2 + m^2}. \tag{61}$$

Let us make some qualitative remarks about the physical meaning of this diagram (and of similar ones). Near the upper limit of integration, the loop momentum is very large, $l \sim \Lambda$, the shift (3) is irrelevant and the integrand is similar to the corresponding one in the full theory. On the contrary, near the lower limit of integration, the loop momentum is very small, $l \sim \Lambda_E$. We have that $l^2/2M \ll l_4$, and the integrand resembles that of the static theory. The integrand in eq. (61) therefore interpolates between the region in momentum space in which the heavy particle H is dynamical and the region in which H is essentially static. The amplitude therefore includes the effects of the fluctuations with momenta both greater and smaller than M. The transformation (7) indeed lowers the cut-off in such a way that we cross a physical threshold, the heavy particle mass.

The self-energy at zero external momentum is given by:

$$\tilde{\Sigma}_H(0) = \frac{\lambda^2}{16\pi^2} 2M \left[\log \frac{\Lambda^2}{M^2} + 1 - 4\frac{\Lambda_E}{M} + O((\Lambda_E/M)^2) \right]. \tag{62}$$

The mass renormalization of the heavy scalar δM is related to the self-energy by:

$$\delta M = -\tilde{\Sigma}_H(0). \tag{63}$$

The first derivative of $\tilde{\Sigma}_H$ with respect to k_4 is given by:

$$\left. \frac{\partial \tilde{\Sigma}_H}{\partial k_4} \right|_{k=0} = -i\frac{\lambda^2}{16\pi^2} 2 \left[\log \frac{M^2}{\Lambda_E^2} - 1 + O(\Lambda_E/M) \right]. \tag{64}$$

In a renormalizable theory (instead of superrenormalizable), we would have an additional term of the form $\log \Lambda^2/M^2$ at the right hand side of eq. (64). In our model, a term of this kind appears only in the mass renormalization, because field and coupling constant corrections are $u.v.$ finite. The physical origin of the two kind of logs will be discussed in sect.4.2.

One of the effects of the integrated modes is that of modifying the normalization of the kinetic operator given in the original hamiltonian. This

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effect is quantified by the Z-factor relating the on-shell renormalized field ϕ_{OS} to the bare field ϕ_B, defined by

$$\phi_{OS} = \frac{\phi_B}{\sqrt{Z}}, \tag{65}$$

and given by:

$$Z = 1 - i\frac{\partial\tilde{\Sigma}_H}{\partial k_4}\Big|_{k=0}. \tag{66}$$

The vertex correction is given by:

$$\tilde{V}(k,k') = \lambda^3 \int \frac{d^4l}{(2\pi)^4} \frac{1}{l^2 + m^2} \times \tag{67}$$

$$\times \frac{1}{k_4 + l_4 - i(k+l)^2/2M} \frac{1}{k'_4 + l_4 - i(k'+l)^2/2M}.$$

At zero external momenta:

$$\tilde{V}(0,0) = -\frac{\lambda^3}{16\pi^2}\left[2\log\frac{M^2}{\Lambda_E^2} + O(\Lambda_E/M)\right]. \tag{68}$$

In a renormalizable theory we would have an additional term of the form $\log\Lambda^2/M^2$ at the right hand side of eq. (68), related to the ultraviolet divergence of the vertex correction.

The bare charge is therefore renormalized as follows:

$$\lambda \rightarrow \lambda\left[1 + \frac{\lambda^2}{16\pi^2}2\log\frac{M^2}{\Lambda_E^2} + O(\Lambda_E/M)\right]Z = \lambda\left[1 + \frac{\lambda^2}{16\pi^2}2 + O(\Lambda_E/M)\right]. \tag{69}$$

The cancellation of the $\log M^2/\Lambda_E^2$ terms is related to the conservation of the current $\phi^\dagger\phi$ in the effective theory and is not specific of our model.

In QED, we would have an additional term of the form $\beta_0\log\Lambda^2/M^2$ at the right hand side of eq. (69). This implies that the (bare) coupling constant of the original theory must be evolved from Λ to M only, because it remains unchanged as we enter the effective theory region (i.e. the cut-off becomes smaller than M).

4.2   Qualitative considerations

Quite generally, we can identify three regions of momenta which must be integrated for the derivation of the effective hamiltonian.

1) The first one involves momenta l between the original cut-off Λ down to a scale of the order of the heavy particle mass:

$$M^2 \ll l^2 \ll \Lambda^2. \tag{70}$$

In this region, the integrand is similar to that one in the full theory, and the mass M of the heavy particle can be neglected. We are basically computing the quantum fluctuations for a relativistic massless particle. The contribution to mass renormalization coming from this region, for example, is given by:

$$\frac{\delta M}{M} \sim \int \frac{d^4 l}{(l^2)^2} \sim \log \frac{\Lambda^2}{M^2}, \tag{71}$$

because the mass of the heavy particle acts as an infrared regulator. It is therefore this region which produces the $\log \Lambda^2 / M^2$ term in the mass renormalization constant of our model, and which would give an analogous term in Z and V for a renormalizable theory. The physical origin of the logarithm is the scale invariance of a massless field theory: there is not any scale entering the integrand of eq. (71), so that every order of magnitude range of energies gives the same contribution to $\delta M / M$. These kind of logs can be resummed with the usual technique of performing many small cut-off lowerings, from Λ to $\Lambda - \delta \Lambda$, from $\Lambda - \delta \Lambda$ to $\Lambda - 2\delta \Lambda$, etc. Region (70) is, let us say, a 'scaling region', because we do not cross any physical threshold and we are basically in a massless theory. This region is unbounded from above and is the one relevant to study the continuum limit of the theory.

2) The second region is the border between the full and the effective theory, and involves momenta of the order of the mass,

$$l^2 \sim M^2. \tag{72}$$

This is not a 'scaling' region because we are crossing a physical threshold, the heavy particle mass. This region does not produce large logs because it does not extend for many orders of magnitude in the energy scale, but gives rise to finite terms in the matching constants. Crossing region 2), physics goes from a relativistic to a non relativistic one: there is no more enough

energy to produce heavy pairs and to give a relativistic momenta to a heavy particle.

3) The third region extends from a scale of the order of the heavy particle mass M, down to the ultraviolet cut-off Λ_E of the effective theory:

$$\Lambda_E^2 \ll l^2 \ll M^2. \tag{73}$$

This region is unbounded from below because Λ_E may be arbitrarily small with respect to M. This is again a 'scaling' region, because the heavy particle mass disappears from the dynamics. We are in a different kind of 'massless' theory with respect to region 1). The contribution of this region to field and vertex renormalization is given by integrals of the form:

$$\int \frac{d^4l}{l_4^2 \, l^2} \ \sim \ \log \frac{M^2}{\Lambda_E^2}. \tag{74}$$

In general, it is this region which produces the logarithms of the form $\log(M^2/\Lambda_E^2)$. The origin of these logs is the scale invariance of a theory describing massless particles interacting with fixed (i.e. infinite mass) sources. There is not any scale entering in the integrand, and the integral is regulated by the heavy particle mass and the effective theory cut-off. These logarithms can be resummed with the usual technique, composing small lowerings of the cut-off of the effective theory. This region has been extensively studied in the previous section.

Up to now we have considered a model with a heavy particle and a massless one, but it is not difficult to extend the above considerations to a model with, for example, two species of heavy particles and a massless particle. Let us assume that the masses of the heavy particles are very different, i.e. $M' \ll M$ (this case presents in heavy flavor physics, when both the beauty and the charm quarks are treated in the effective theory and their ratio M_b/M_c is considered large). In this case, there are two others relevant regions to integrate away to derive the effective theory. The first one lies between 1) and 2), and consists of momenta between M and M',

$$M'^2 \ll l^2 \ll M^2. \tag{75}$$

In this region the heavier particle behaves as a static particle, while the lighter one behaves as a massless particle. This is a 'scaling region'. The

contributions to the renormalization constants from this region are therefore of the form:

$$\int \frac{d^4l}{l_4^2 \, l^2} \sim \log \frac{M^2}{M'^2},$$
(76)

because the larger mass M acts as an ultraviolet cut-off to the integral while the smaller mass M' acts as an infrared cut-off. A log like that in eq. (76) is usually called an hybrid log [4].

The second region is the border between the full theory for the lighter particle and the effective one. It consists of momenta l of the order:

$$l^2 \sim M'^2.$$
(77)

We can extend this analysis to a theory with an arbitrary number of heavy particles, with masses $...M'' \ll M' \ll M$. In every interval between two masses, we have a scaling region which gives rise to a log of the mass ratio, and a boundary region.

4.3   Connection with usual matching theory

Let us discuss now the connection between our formulation of the effective hamiltonians and the usual matching theory.

Consistency requires that the amplitudes of the static theory coincide with those of the original (full) theory at the lowest order in $1/M$. To this end, renormalization constants for the effective theory operators are introduced, which are determined by a set of matching conditions. Let us consider a specific example, matching of the heavy scalar propagator. The bare propagators of the full and the static theory are given, near the mass-shell, respectively by:

$$i\Delta_F(k) = \frac{-i}{k_4[1 + i(\partial\Sigma_F/\partial k_4)_0] + O(1/M)}$$
(78)

$$i\Delta_S(k) = \frac{-i}{k_4[1 + i(\partial\Sigma_S/\partial k_4)_0] - i\epsilon},$$

where:

$$\left(\frac{\partial\Sigma_F}{\partial k_4}\right)_0 = -2i\lambda^2 \int_0^\Lambda \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2)^2} \frac{l_4}{l_4 - il^2/2M}$$
(79)

$$\left(\frac{\partial\Sigma_S}{\partial k_4}\right)_0 = -2i\lambda^2 \int_0^{\Lambda_E} \frac{d^4l}{(2\pi)^4} \frac{1}{(l^2)^2}.$$

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To avoid cumbersome expressions, mass terms have been omitted in eqs. (78). The matching constant Z_m is defined requiring that the static propagator coincides with the full-one at the lowest order in $1/M$:

$$i\Delta_F(k) = Z_m\, i\Delta_S(k). \tag{80}$$

Using eqs. (78) and eq. (80), we derive the following expression for the matching constant:

$$Z_m = 1 - i\left[\left(\frac{\partial\Sigma_F}{\partial k_4}\right)_0 - \left(\frac{\partial\Sigma_S}{\partial k_4}\right)_0\right]. \tag{81}$$

Let us consider now the renormalization of the heavy particle field induced by our transformation. After the functional integration has been done, the bare propagator of the heavy scalar is given, near the mass-shell, by:

$$i\Delta(k) = \frac{-iZ}{k_4 - i\epsilon + O(1/M)}. \tag{82}$$

where Z has been defined in eq. (65).

The renormalization constant Z takes into account the modification of the field normalization as we scale down the cut-off from Λ to Λ_E. This factor therefore includes the effects of the fluctuations with momenta between Λ and Λ_E in the normalization of the field, and must be identified with the matching constant Z_m:

$$Z = Z_m. \tag{83}$$

Using eqs. (64), (66), (79) and (81), we see that the two matching constants do not coincide completely, and the difference is given by:

$$\begin{aligned}
Z_m - Z &= -i\left[\left(\frac{\partial\Sigma_F}{\partial k_4}\right)_0 - \left(\frac{\partial\Sigma_S}{\partial k_4}\right)_0 - \left(\frac{\partial\tilde{\Sigma}}{\partial k_4}\right)_0\right] \\
&= -2\lambda^2 \int_0^{\Lambda_E} \frac{d^4k}{(2\pi)^4} \frac{1}{(l^2)^2}\left[\frac{l_4}{l_4 - il^2/2M} - 1\right].
\end{aligned} \tag{84}$$

Expanding in powers of $1/M$ the heavy scalar propagator in eq. (84), we see that $Z_m - Z$ contains terms of the form $(\Lambda_E/M)^n \ll 1$ with $n \geq 1$. Z_M and Z therefore coincide at the lowest order in $1/M$. We believe that this is true at every order in perturbation theory. The same result holds for all the other Green functions, i.e. differences in the matching constants arise only at order

$1/M$. The static theory is constructed dropping $1/M$ operators as well as corrections of order $1/M$ to the matching constants of the static operators. We conclude therefore that the transformation (7) reproduces correctly the matching constants of the static theory.

5 Conclusions

We have formulated the effective theory for heavy particles with renormalization group techniques, by integrating away the states with high velocity and with high virtuality. The fixed point hamiltonian of our transformation coincides with the static hamiltonian, and irrelevant operators are identified with $1/M$ corrections. Matching conditions between the full and the static theory have not to be imposed by hand as in the usual approach, but come out quite naturally as dynamical effects of the RG flow.

Our interest was to relate, from first principles, quantum field theory, a system with an infinite number of degrees of freedom, with quantum mechanics.

The formulation of the effective theories for heavy particles presented is non-perturbative, just like Wilson's transformation. The prescription for the degrees of freedom to integrate away and the equations for the RG flow indeed do not rely on any perturbative expansion. Our transformation can accomodate, for example, instanton effects [13], or can be computed exactly with numerical methods. Whenever we apply perturbation theory to it, we are able to reproduce all the know results for the matching constants.

While this approach does not provide us with new computational tools, we think that it clarifies the physical content of the effective theories for heavy particles, and constitutes a solid framework for (future) non-perturbative computations.

Acknowledgments

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Figure Captions

Fig.1: Momentum space of the heavy particle. The states with momenta inside the white region are integrated away, while the states with momenta inside the black sphere are not integrated and constitute the dynamical variables of the effective action for heavy particles. The black sphere is centered around the momentum of a particle at rest, $r = (M, \vec{0})$, and has a radius $\Lambda_E \ll M$.

Fig.2: Momentum space of the light particle. The states with momenta inside the white region are integrated away, while those one inside the black sphere are not integrated and constitute the dynamical variables of the effective action for light particles. The black sphere is centered around the null momentum and has a radius $\Lambda_E \ll \Lambda$.

Fig.3: One-loop self-energy diagram of the heavy scalar. The continuous line represents the heavy particle, while the dotted line the light scalar.

Fig.4: One-loop correction of the heavy-heavy-light vertex.

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